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Rosters and connected apportionments

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Abstract
Affirmative action in India reserves explicit proportions of seats and jobs in publicly funded institutions for various beneficiary groups. Because seats are indivisible and arise in small numbers over time, implementation of this policy requires that beneficiary groups take turns claiming seats, for which purpose India relies on a device called a roster. We study the problem of constructing a roster, which involves addressing a series of connected apportionment problems. To identify suitable apportionment methods, six essential requirements direct our search to a large class of divisor methods. We show that the Webster–Sainte-Laguë method is the unique divisor method that satisfies several practical properties and fairness criteria. Comparative analysis between an existing Indian roster and the application of the Webster–Sainte-Laguë method highlights that method's benefits.

Keywords Rotation · Indivisibility · Apportionment · Divisor methods

1 Introduction

Positive discrimination policies, such as affirmative action in the U.S. and reservation policies in India, are similar in motivation, but they diverge significantly in their implementations. Unlike affirmative action in the US, the Indian reservation policy explicitly prescribes proportions of seats and jobs in publicly funded institutions to various beneficiary groups (Weisskopf, 2004). Every recruitment or admissions advertisement must include information about the proportion of government positions that are specifically designated as “reserved" for various protected groups. Since seats are indivisible and arise in small numbers over time, innovative methods are used to help achieve the objectives of the reservation policy in practice. For instance, a university may appoint at most one assistant economics professor every year, while the reservation policy may have five beneficiary groups.

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To ensure that, over a period, each beneficiary group receives its reservation policy-prescribed percentage of seats, India devised a tool called a *roster*.[1]

The publicly announced *roster* details a 200-item-long sequence in which beneficiary groups of a reservation policy take turns claiming seats. The objective of maintaining a roster is to provide representation in proportion to the reservation fractions mandated by the associated policy, and the chosen route involves the rotation for claiming seats[2]:

> Though members of a particular category in a particular recruitment year may be unlucky and may not get proportionate benefit but their lucky successors in later recruitment years may get more than what is due to them, thus, making up for the earlier deficiency and vice versa.

Since 1997, all public institutions in India, including universities and hospitals, have been required to maintain vacancy-based rosters to implement the imposed reservation policy.[3] The use of publicly declared rosters is greatly valued by legislators and their electorates due to their ease, transparency, and credibility in terms of implementing the reservation policy. However, despite their importance, no formal scrutiny or advice is currently available to inform the designs of these rosters. This study represents the first attempt to examine and analyze the designs of such rosters.

A reservation policy dictates the total number of turns on a roster received by a beneficiary group. However, the turn sequence is not fixed by legislation and is therefore up to the designers (which are the many state governments of India). Since seats arrive over time in small numbers, the delay observed when claiming seats occurs naturally. However, it would be an unjustified layer of partiality for the delay to be systematically associated with a beneficiary group's proportion as mandated by the policy.

In this note, we study the problem of constructing rosters. Section 2 presents rosters as solutions to a series of connected apportionment problems. Section 3 shows that any serious contender for solving those problems among the available apportionment methods must be derived from a large class of divisor methods. Section 4 shows that practical and fairness considerations favor the Webster–Sainte-Laguë method among all divisor methods. Furthermore, Theorems 2 and 3 give two new properties of the Webster–Sainte-Laguë method. Section 5 scrutinizes the Indian roster while contrasting it with the roster produced by the Webster–Sainte-Laguë method. Section 6 concludes. The proofs are relegated to Appendix 1.

2 Formulation

A roster construction problem is a tuple $\Lambda = (\mathcal{C}, \boldsymbol{\alpha}, n)$. $\mathcal{C}$ is a finite set of categories, where $m := |\mathcal{C}| \geq 2$. A reservation policy is defined by a vector of fractions $\boldsymbol{\alpha} = (\alpha_j)_{j\in\mathcal{C}}$. For each category $j \in \mathcal{C}$, $\alpha_j \in (0,1)$ fractions of turns are reserved so that $\sum_{j\in\mathcal{C}} \alpha_j = 1$. The size of the roster is n, where n is a positive integer. Throughout the paper, we fix a set of categories $\mathcal{C}$ and a reservation policy $\boldsymbol{\alpha}$.

[1] For India's roster, visit https://dopt.gov.in/sites/default/files/ewsf28fT.PDF, which was last accessed on 24 May 2024.

[2] See page 10 of https://districts.ecourts.gov.in/sites/default/files/RESERVATION%20RULES%20IN%20GOVT%20SERVICES%20RAJASTHAN, which was last accessed on 24 May 2024.

[3] See https://documents.doptcirculars.nic.in/D2/D02adm/36012_2_96_Estt(Res), which was last accessed on 24 May 2024.

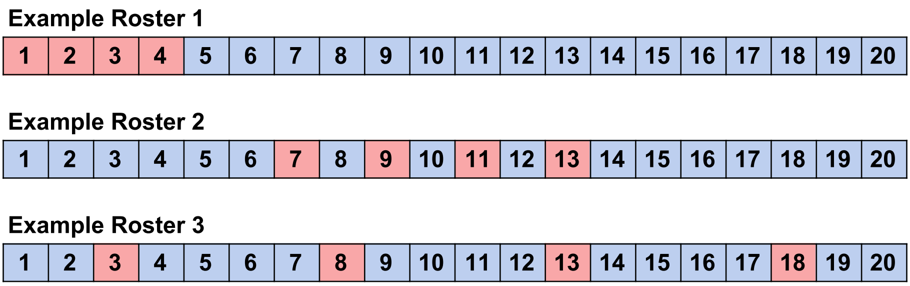


Fig. 1 Example rosters

A roster $R_n : \{1, \cdots, n\} \to \mathcal{C}$ maps each position to a category such that $|R_n^{-1}(j)| = \alpha_j n$ for all $j \in \mathcal{C}$. We denote the set of possible rosters by $\mathcal{R}_n$. The definition incorporates the idea that the total number of positions a roster assigns to a category is the same as the proportion given by the reservation policy; that is, all categories must obtain their quantity of reserved positions once n positions are filled. Therefore, we can restrict our attention to only those rosters where this is possible.

We denote by x_j^t the number of seats given to category j until position t under roster R_n; that is, $x_j^t := |\{s \in R_n^{-1}(j) \mid s \leq t\}|$. We denote by n_j the total number of seats given to category j; that is, $n_j := \alpha_j n$. In line with the practice of making rosters, we assume that n is chosen such that $n_j \in \mathbb{N}$; that is, the total number of turns for each category is a natural number.

We next introduce an example that makes the notion of a roster easier to comprehend. Two categories are utilized to provide an easy illustration. This example is also sufficient for presenting the various aspects of designing rosters in the upcoming sections.

Example 1 Consider a problem $\Lambda = (\{R, B\}, \boldsymbol{\alpha} = [0.2, 0.8], 20)$. Two categories $\mathcal{C} = \{R, B\}$ are represented by red and blue colors. The reservation policy reserves 20% of the positions for members of category R. The size of the roster is $n = 20$. Therefore, the numbers of positions assigned to categories R and B are 4 and 16, respectively. Figure 1 illustrates three possible rosters for the problem. For instance, Example Roster 1 is

$$R_n(k) = \begin{cases} R, & \text{if } k \in \{1, 2, 3, 4\} \\ B, & \text{otherwise} \end{cases}$$

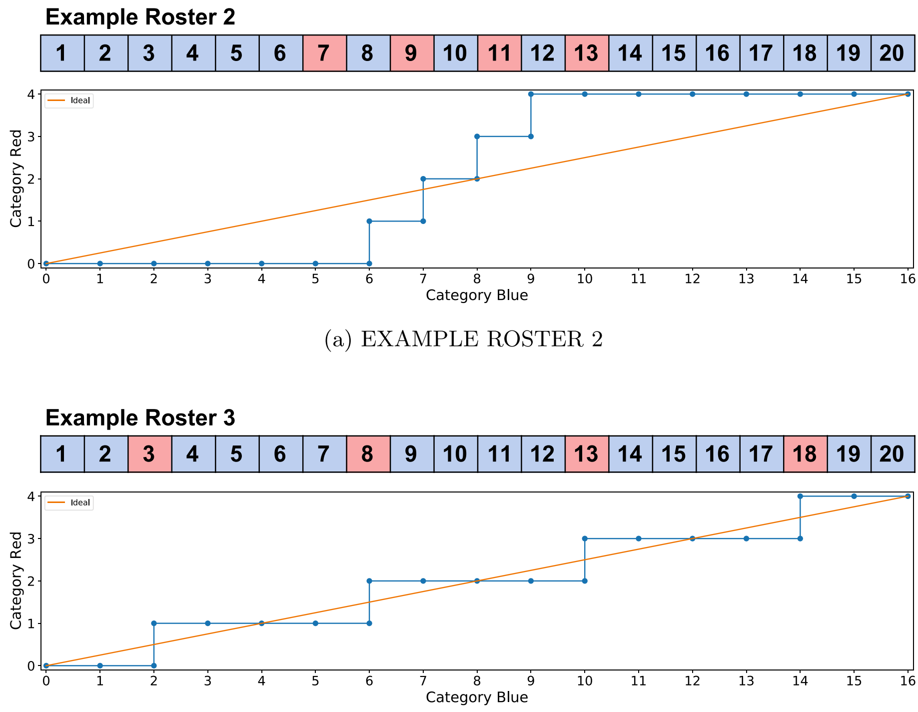
The following representation of a roster entails a staircase where each step is the turn of one category. The staircase representation of the roster R_n is

$$\mathbf{x^t} = (x_1^t, \cdots, x_j^t, \cdots, x_m^t), \quad \text{for} \quad t \in \{1, \cdots, n\}.$$

Fig. 2 illustrates the staircase representation for the two rosters depicted in Fig. 1.[4]

The standard unit vector in the direction of the j-th axis is denoted by $\mathbf{e_j}$; that is, in this vector, the j-th component equals 1, and all other components equal 0. When given two consecutive points $\mathbf{x^{t-1}}$ and $\mathbf{x^t}$, if $\mathbf{x^t} = \mathbf{x^{t-1}} + \mathbf{e_j}$, we say that the staircase moves in

[4] The staircase representation bears a resemblance to step-based indices, which were introduced and characterized byChambers and Miller (2014), that prioritize citations and publications to determine the productivity or influence of a scientist. We thank an anonymous referee for pointing out this connection.



(a) EXAMPLE ROSTER 2

(b) EXAMPLE ROSTER 3

Fig. 2 Staircase representation

direction j at step t; that is, $x_j^t = x_j^{t-1} + 1$. Note that for the staircase representation of a roster, $\mathbf{x^t}$ can move in only one direction at any step.

The roster construction problem may therefore be seen as a series of connected apportionment problems, where each step t of the staircase $\mathbf{x^t} = (x_1^t, \cdots, x_m^t)$ is an apportionment of t seats among m categories with $\mathbf{q^t} = (q_j^t)_{j \in C} := (\alpha_1 t, \cdots, \alpha_m t)$ claims. Note that step t corresponds to the *house size* and that claim q_j^t corresponds to the *quota* in the original apportionment model (see Balinski & Young 1982).

3 Methods

A method of apportionment is a point-to-set mapping Φ that assigns at least one solution $\mathbf{x^t}$ to each $\mathbf{q^t}$. Numerous methods have been proposed and utilized to address this problem (see Balinski & Young, 1994; Pukelsheim, 1982; Young, 2017). To grasp the characteristics of their solutions, examining the properties to which they adhere is crucial. Among these properties, three are unquestionably essential and have been satisfied by every method ever seriously proposed.

- A method of apportionment Φ is anonymous if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ implies that $\sigma(\mathbf{x^t}) \in \Phi(\sigma(\mathbf{q^t}))$ for any permutation σ of the involved categories. A category should receive the same number of seats wherever it appears in the list of categories.
- A method of apportionment Φ is exact if $\mathbf{q^t}$ is integer-valued, which implies that $\Phi(\mathbf{q^t}) = \mathbf{x^t}$. If perfect proportionality can be achieved, it must be achieved.
- A method of apportionment Φ is responsive if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ and $q_i^t > q_j^t$ implies that $x_i^t \geq x_j^t$. A category with a higher claim should never receive fewer seats.

The basic principle of fair apportionment (Balinski, 2005) is that "any part of a fair apportionment must be fair." To define this idea, let $\mathbf{x}_S^t := \left(x_j^t\right)_{j\in S}$, $\mathbf{q}_S^t := \left(q_j^t\right)_{j\in S}$ and $x(S) := \sum_{j\in S} x_j^t$ for $S \subset \mathcal{C}$.

- A method of apportionment Φ is consistent if when $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$, $\mathbf{x}_S^t \in \Phi\left(\mathbf{q}_S^{x(S)}\right)$ for any subset of categories $S \subset \mathcal{C}$; moreover, if a subproblem has another solution $\mathbf{z}_S^{x(S)} \in \Phi\left(\mathbf{q}_S^{x(S)}\right)$, another solution to the problem itself exists: $\left(\mathbf{z}_S^{x(S)}, \mathbf{x}_{(\mathcal{C}\setminus S)}^t\right) \in \Phi(\mathbf{q}^t)$.

Two other natural properties for apportionment methods that can solve roster construction problems are as follows.

- A method of apportionment Φ is house-monotonic if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ implies that there is some $\mathbf{x^{t+1}} \in \Phi(\mathbf{q^{t+1}})$ for which $\mathbf{x^{t+1}} \geq \mathbf{x^t}$. Reservations are irreversible, going from step t to step $t+1$ in a roster, and the number of positions each category receives up until each step can only weakly increase.
- A method of apportionment Φ is balanced if $\mathbf{x^t} \in \Phi(\mathbf{q^t})$ and $q_i^t = q_j^t$ implies that $|x_i^t - x_j^t| \leq 1$. If two categories have the same reservation fractions and therefore the same claims, their apportionments should not differ by more than one seat.

Following Balinski and Ramirez (2014), a divisor function d is any monotonic real-valued function defined on the nonnegative integers satisfying $d(k) \in [k, k+1]$ for any integer k, where there exists no pair of integers $a \geq 0$ and $b \geq 1$ with $d(a) = a+1$ and $d(b) = b$.

The divisor method Φ^d based on d is

$$\Phi^d(\mathbf{q^t}) = \left\{ \mathbf{x^t} \,:\, \min_{x_i^t>0} \frac{q_i^t}{d\left(x_i^t - 1\right)} \geq \max_{x_j^t\geq 0} \frac{q_j^t}{d\left(x_j^t\right)}, x(\mathcal{C}) = \sum_{j\in\mathcal{C}} x_j^t = t \right\}.$$

For divisor methods, the values ${}^{q_i^t}/{}_{d(x_i^t)}$ for all i and all integers x_i^t give priority (in decreasing order) to the category i that receives its x_i^tth seat. Any divisor method is consistent, house-monotonic, and balanced in addition to satisfying the three essential properties (see Palomares et al., 2017; Pukelsheim, 2022). Arguably, the most important result in the theory of apportionment that characterizes divisor methods provides further affirmations.

Theorem 1 (Characterization of divisor methods (Balinski & Young, 1982, p. 147)) *A method of apportionment Φ is consistent, responsive, exact, and anonymous if and only if it is a divisor method Φ^d.*

4 Why Webster–Sainte-Laguë?

The gist of the previous section is that any method of apportionment that is seriously worthy of investigation must be a divisor method. The following question then arises: Which of the infinite number of divisor methods should be chosen?

This section contends that one particular member stands out, the Webster–Sainte-Laguë method, which requires $d(a) = a + \frac{1}{2}$ (also known as the Sainte-Laguë method or the major fractions method).

A roster R_n is a Webster–Sainte-Laguë staircase if for all $\mathbf{x^t}$ and $t \in \{1, \cdots, n\}$ we have

$$\min_{x_i^t > 0} \frac{\alpha_i}{x_i^t - 0.5} \geq \max_{x_j^t \geq 0} \frac{\alpha_j}{x_j^t + 0.5},$$

which is the min-max inequality that characterizes Webster–Sainte-Laguë apportionments.

4.1 Concatenation invariance

The following two principles are followed for the maintenance of rosters.[5]

(f) The register/roster register shall be maintained in the form of a running account year after year. For example if recruitment in a year stops at point 6, recruitment in the following year shall begin from point 7.

(h) In case of cadres where reservation is given by rotation, fresh cycle of roster shall be started after completion of all the points in the roster.

Therefore, the roster decides not only the allocation of seats $1, 2, \cdots, n$ but also the allocation of seats $n+1, \cdots, 2n$ and $2n+1, \cdots, 3n$. A roster allocates an infinite sequence of seats constructed as a concatenation of infinitely many finite seat sequences of length n. Our next property requires the selected apportionment method to be invariant to such concatenation operations. That is, a roster of size kn can be constructed by concatenating k copies of a roster of size n. For example, among the rosters depicted in Fig. 1, only example roster 3 is invariant to such a concatenation process.

Given a roster construction problem Λ, let s denote the size of the smallest roster possible; that is, let s be the lowest common denominator of the reservation fractions.

- A method of apportionment Φ is concatenation invariant if $t \in \mathbb{N}_+$, $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$, and $\mathbf{x}^s \in \Phi(\mathbf{q}^s)$ implies $\mathbf{x}^t + \mathbf{x^s} \in \Phi\left(\mathbf{q}^{t+s}\right)$. Note that $\mathbf{q}^{t+s} = \mathbf{q}^t + \mathbf{q}^s$.

Not all divisor methods are concatenation invariant, but the Webster–Sainte-Laguë method is,[6] More generally, the following simpler and more manageable class of divisor methods, known as parametric methods, are concatenation invariant. The Adams,

[5] See page 1 of https://dopt.gov.in/sites/default/files/Ch-05_2014, which was last accessed on 24 May 2024.

[6] An example of a divisor method that is not concatenation invariant is Hill's method $d(a) = \sqrt{a(a+1)}$. Consider example 1, with $\Lambda = (\{R, B\}\boldsymbol{\alpha} = [0.2, 0.8], 20)$, and notice that $s = 5$. To see why Hill's method is not concatenation invariant, consider $t = 2$. $(1,1) \in \Phi((0.4, 1.6))$, and $(1,4) \in \Phi((1,4))$, but $(2,5) \notin \Phi((1.4, 5.6))$; instead, only $(1,6) \in \Phi((1.4, 5.6))$.

Webster–Sainte-Laguë and Jefferson methods are parametric methods, but Dean's and Hill's methods are not.

A parametric method ϕ^{δ} is a divisor method Φ^{d} based on $d(k) = k + \delta$, where $0 \leq \delta \leq 1$, for all integers $k \geq 0$.

Theorem 2 *All parametric methods are concatenation-invariant divisor methods.*

Corollary 1 *The Webster–Sainte-Laguë method is a concatenation-invariant divisor method.*

Apportionment problems form a subclass of a much larger class of *claims problems*, which were recently reviewed in Thomson (2019), where the allocations are not restricted to natural numbers. Parametric methods, which were introduced and characterized by Young (1987), form an important class of rules for claims problems. Kaminski (2000) described and characterized them using a 'hydraulic' metaphor. Kaminski (2006) generalized Young's characterization to separable type spaces. Moreno-Ternero and Roemer (2006) adapted this characterization to a model in which claims are replaced by utility functions. Chambers and Moreno-Ternero (2017) characterized the parametric methods that satisfy the additional *composition down* requirement.

4.2 Equitable treatment in a roster

Since a roster entails a series of connected apportionments, the set of positions at which a category is at an advantage and the distribution of these sets must be further analyzed to determine the goodness of various methods and to differentiate or classify them. In addition to the previous argument in favor of using the Webster–Sainte-Laguë method, we next argue that the rosters generated solely through the application of the Webster–Sainte-Laguë method treat beneficiaries in an equitable manner.

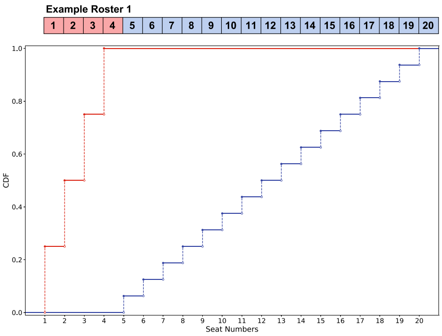
Since seats are indivisible, at every position in a roster, there will always be a certain partiality between any two categories that gives one of the categories a slight advantage over the other. The distribution of seats for a category would make this advantage clear in the case of a roster. The cumulative distribution of seats for category j under roster R_n is

$$F_j(t) := \frac{|\{s \in R_n^{-1}(j) \mid s \leq t\}|}{|R_n^{-1}(j)|} = \frac{x_j^t}{n_j}, \quad \text{for} \quad t \in \{1, \cdots .n\}.$$

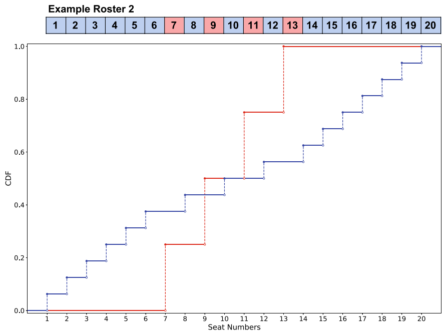
These cumulative distribution functions measure the fraction of seats a category receives until position t, that is, the number of seats given to a category until position t over the total number of seats given to a category. For instance, Fig. 3 illustrates the cumulative distributions of seats for the rosters depicted in Fig. 1.

4.2.1 Spreading seats as evenly as possible

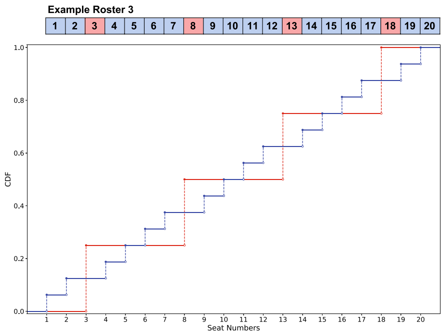
Had the seats been divisible, a uniform distribution would have been the ideal seat allocation for the equitable treatment of the different categories. In that case, the seats would have been spread "as evenly as possible" without favoring any category over the other at any point in the roster, thus treating all categories as equally as possible. However, since seats are indivisible, at each step, some categories are overrepresented, while



(a) EXAMPLE ROSTER 1



(b) EXAMPLE ROSTER 2



(c) EXAMPLE ROSTER 3

Fig. 3 Distribution of seats

others are underrepresented. As a measure of the grievances of each category, deviations from the uniform distribution form a reasonable measure of partiality.

We denote by $U(t)$ the uniform distribution; that is, for any $t \in \{1, \cdots, n\}$, $U(t) = t/n$. When evaluating the nonuniformity of a roster at position t, we consider $|F_j(t) - U(t)|$ to be the distance between the distribution of seats for category j and the uniform distribution; this distribution is squared, weighted by the claim α_j, and summed across categories. This leads to a nonuniformity index at step t, which is defined as

$$\mathrm{DI}(\mathbf{x^t}) = \sum_{j \in \mathcal{C}} \alpha_j (F_j(t) - U(t))^2.$$

- A method of apportionment Φ minimizes the nonuniformity index DI if $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$ implies that $\mathrm{DI}(\mathbf{x^t}) \leq \mathrm{DI}(\mathbf{x^t} + \mathbf{e_j} - \mathbf{e_i})$ for all categories $i, j \in \mathcal{C}$.

Theorem 3 *The Webster–Sainte-Laguë method is the unique divisor method that minimizes the nonuniformity index DI.*

Uniformity hints at proportionality, and the Webster–Sainte-Laguë method of apportionment hints at a Sainte-Laguë index of proportionality (Lijphart & Gibberd, 1977, p. 241). We next show how these two factors are related.

Consider the ratio between the seat allocation x_j^t and claim q_j^t for each category j and position t, in particular, ${x_j^t}/{q_j^t}$. For a perfectly proportional outcome, ${x_j^t}/{q_j^t} = 1$ for each j and t. When evaluating the disproportionality of a roster at position t, we consider $\left|\frac{x_j^t}{q_j^t} - 1\right|$ as an error term for each category j; this term is squared, weighted by the claim q_j^t, and summed across categories. This leads to the well-known Sainte-Laguë index at step t, which is defined as

$$\mathrm{SLI}(\mathbf{x^t}) = \sum_{j \in C} q_j^t \left(\frac{x_j^t}{q_j^t} - 1 \right)^2 = \sum_{j \in C} \frac{(x_j^t - q_j^t)^2}{q_j^t}.$$

It can easily be shown that the nonuniformity index and the Sainte-Laguë index of disproportionality are comonotonic. In particular, we have the following relationship.

Proposition 1 $DI(\mathbf{x^t}) = {t}/{n^2}\, SLI(\mathbf{x^t})$.

4.2.2 Minimizing inequality

Since seats are indivisible, at every position in the roster, there will always be a certain partiality between any two categories that gives one of the categories a slight advantage over the other. It is straightforward to say that for any pair of categories $i, j \in \mathcal{C}$, category i is favored relative to category j at position t under roster R_n if $F_i(t) > F_j(t)$. One measure of inequality therefore is $|F_i(t) - F_j(t)|$. For example, in Fig. 3, Example Roster 3 has less inequality than does Example Roster 2 at all positions. Huntington (1928) wrote that whether “such a transfer should be made or not depends on whether the amount of inequality between the two states after the transfer is less or greater than it was before; if the amount of inequality is reduced by the transfer, it is obvious that the transfer should be made.”

- A method of apportionment Φ minimizes inequality if $\mathbf{x^t} \in \Phi\left(\mathbf{q^t}\right)$ implies that $\left|\frac{x_i^t}{n_i} - \frac{x_j^t}{n_j}\right| \le \left|\frac{(x_j^t+1)}{n_j} - \frac{(x_i^t-1)}{n_i}\right|$ for all categories $i, j \in \mathcal{C}$.

Theorem 4 (Characterization, Webster–Sainte-Laguë method (Huntington, 1928, p 91)) *The Webster–Sainte-Laguë method is the unique divisor method that minimizes inequality.*

Huntington (1928) referred to the Webster–Sainte-Laguë method as the “method of major fractions.” Aanund Hylland explained this alternative name in Hylland (1990) (see p. 43). A simple proof of this result is available in Balinski and Young (1982) (see p. 101).

4.2.3 Staying near the quota

In line with the concept of Pareto optimality, at any given stage in the roster, it should be impossible to transfer a seat from one category to another in a way that brings the seat allocations of both categories closer to their respective claims. Put simply, it is not feasible to bring one category closer to its claim without simultaneously moving another category further away from its claim.

- A method of apportionment Φ stays near the quota if $\mathbf{x}^t \in \Phi(\mathbf{q}^t)$ implies that there are no categories $i, j \in \mathcal{C}$ such that $|(x_i^t - 1) - q_i^t| < |x_i^t - q_i^t|$ and $|(x_j^t + 1) - q_j^t| < |x_j^t - q_j^t|$.

Theorem 5 (Characterization of the Webster–Sainte-Laguë method (Balinski & Young, 1977, p. 453)) *The Webster–Sainte-Laguë method is the unique divisor method that stays near the imposed quota.*

This result originally appeared in Balinski and Young (1977) for an equivalent concept called “relatively well-roundedness” and what Birkhoff (1976) called “binary consistency.” An alternative proof is available in Balinski and Young (1982) (see p. 132).

5 The roster made in India

$\Lambda^{\mathrm{IN}} = (\{UR, OBC, SC, EWS, ST\}, \boldsymbol{\alpha} = [0.405, 0.27, 0.15, 0.10, 0.075], 200)$ describes the Indian roster construction problem. It consists of five categories of seats: unreserved (UR), other backward classes (OBC), scheduled castes (SC), economically weaker sections (EWS), and scheduled tribes (ST). The reservation policy dictates the division of the 200 seats in a roster among the five categories: 81 for *UR*, 54 for *OBC*, 30 for *SC*, 20 for *EWS*, and 15 for *ST*. However, the positions each category is assigned in a roster are left up to the designer, in this case, the Ministry of Personnel. In all central government institutions, rosters are constructed and maintained as per the most recent revision detailed in *Office Memorandum no. 36039/1/2019-Estt (Res)* dated January 31, 2019, which was issued by the Department of Personnel and Training (Ministry of Personnel, Public Grievances, and Pensions, Government of India),[7]

It is difficult to improve the transparency rosters provided in the nationwide implementation of the reservation policy. Moreover, they do achieve the goal of reserving seats in a manner such that each institution satisfies the prescribed percentage of reserved seats over a sufficiently long period. However, the selected roster construction method can and has been scrutinized due to the delay associated with the arrival of reserved seats. Gupta (2018) criticized the current roster method for delaying reserved seats, which leads to sparse representations of some reserved category candidates. They wrote the following:

> A mathematical juggling has been used by policymakers to reduce the constitutionally mandated reservation for the deprived sections.

[7] For *Office Memorandum no. 36039/1/2019-Estt (Res)* visit https://dopt.gov.in/sites/default/files/ewsf28fT, which was last accessed on 24 May 2024.

Table 1 Instances of pairwise bias

	ST versus EWS	ST versus SC	ST versus OBC	ST versus UR	EWS versus SC	EWS versus OBC	EWS versus UR	SC versus OBC	SC versus UR	OBC versus UR
Indian roster	57	100	146	197	54	126	195	86	198	199
Webster–Sainte-Laguë staircase	3	0	3	3	− 4	2	1	6	9	− 3

5.1 Empirical evaluation

The complaint becomes apparent when analyzing how the distribution of seats is systematically associated with the reservation fraction of a beneficiary group. Recall that for any pair of categories $i, j \in \mathcal{C}$, category i is favored relative to category j at position t under roster R_n if $F_i(t) > F_j(t)$. One measure of partiality is to count such instances. Let $\#|F_i(t) > F_j(t)| := |\{t \in \{1, \cdots, n\} | F_i(t) > F_j(t)\}|$ denote the number of positions at which category i is favored relative to category j. In Table 1, for $\alpha_i < \alpha_j$, we denote the value of $\#|F_i(t) < F_j(t)| - \#|F_i(t) > F_j(t)|$ as the pairwise bias. In pairwise comparisons, this value measures whether the roster tends to exhibit a greater frequency of instances in which it favors categories with larger proportions over the others, surpassing the instances in which the opposite situation occurs.

Table 1 shows the systematic association between the distribution of seats and the reservation fractions. (1) The Indian roster favors categories with larger reservation fractions relative to those with smaller fractions, and (2) the larger the difference between the reservation fractions of two categories is, the greater the associated pairwise bias. For instance, category ST ($\alpha_{ST} = 0.075$), which has the smallest reservation fraction, is compared with categories EWS ($\alpha_{EWS} = 0.1$) and UR ($\alpha_{UR} = 0.405$), which have the largest reservation fractions. The pairwise biases are 57 for ST vs. EWS and 197 for ST vs. UR. Table 1 also shows that such a systematic delay in the arrival of seats for categories with smaller reservation fractions does not arise under Webster–Sainte-Laguë's staircase roster.

In a Webster–Sainte-Laguë staircase, seats are spread "as evenly as possible" without favoring any category at a majority of the points contained in the roster, thus treating all categories as equally as possible. This is shown in Fig. 4, which plots the cumulative seat distribution under the Webster–Sainte-Laguë staircase roster next to that of the current Indian roster. Webster–Sainte-Laguë's staircase roster visibly minimizes nonuniformity. Concurrently, the Indian roster favors categories with larger proportions of seats over those with smaller proportions. For instance, category 1 (UR), which has the largest proportion of seats ($\alpha_{UR} = 0.405$), is compared with category 5 (ST), which has the smallest proportion of seats ($\alpha_{ST} = 0.075$). Throughout the roster, $F_1(t) > F_5(t)$. Similar comparisons can be made between all pairs of categories.

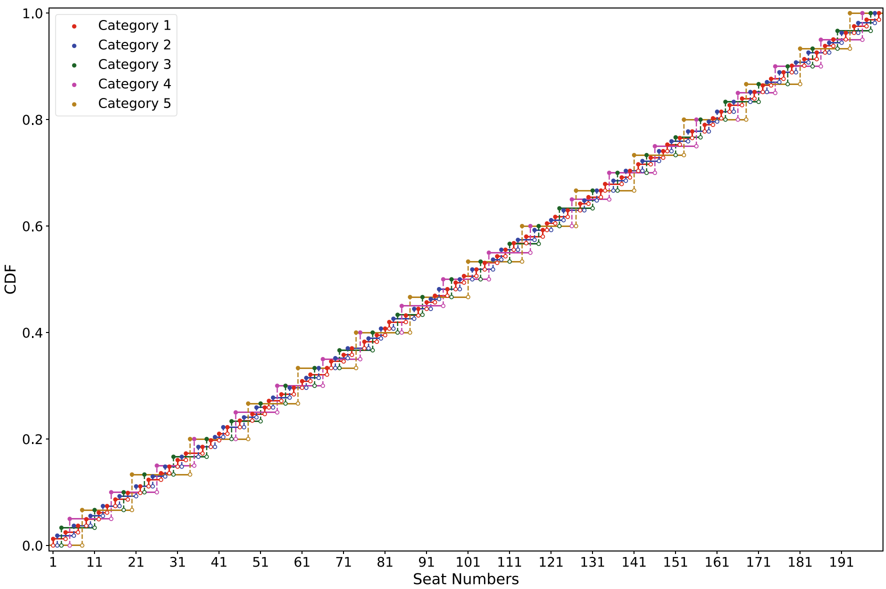
5.2 Theoretical evaluation

The method of apportionment $\Phi^{\mathtt{IN}}$ behind the Indian roster can be summarized as a two-step procedure.[8] First, it allocates claim floors to the OBC, SC, EWS, and ST categories. Then, it allocates the leftover seats to UR. Suppose that more than one category among OBC, SC, EWS, or ST qualifies for an additional seat when moving from step $t-1$ to step t. In this case, a tiebreaker is used to determine the allocation of the tth seat, and the $t+1$th seat is assigned to the other qualifying category. This ensures that the number of positions each category receives up until each step can only weakly increase. Therefore, $\Phi^{\mathtt{IN}}$ assigns solution $\mathbf{x^t}$ to each $\mathbf{q^t}$ as follows:

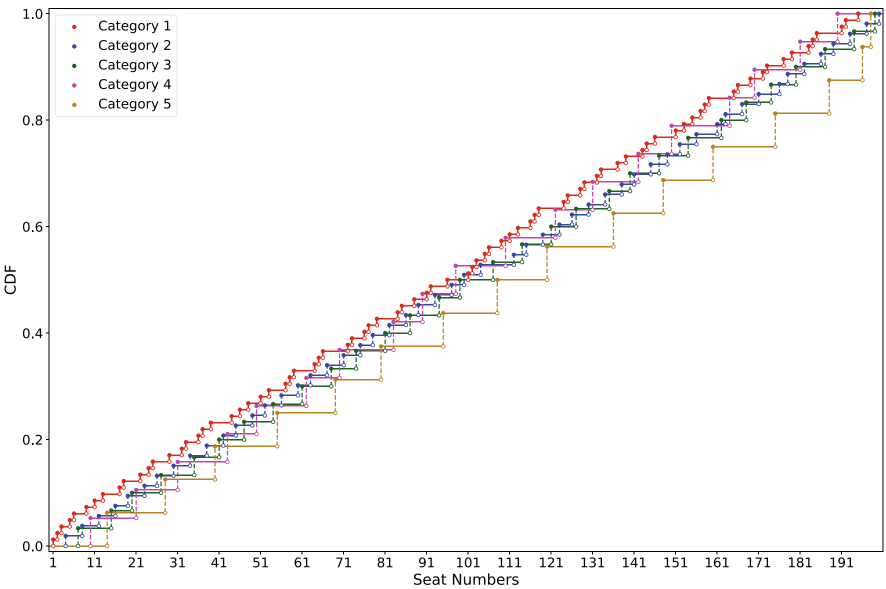
Step 1 For at most one $i \in \{OBC, SC, EWS, ST\}$: $x_i^t = x_i^{t-1} + 1$ if $\lfloor q_i^t \rfloor - x_i^{t-1} \neq 0$.

Step 2 $x_{UR}^t = t - \sum_{i \neq UR} x_i^t$.

[8] The details of the method for making rosters can be found in the *Annexure I to Office Memorandum no. 36012/2/96-Estt(Res)* dated July 2, 1997. Visit https://documents.doptcirculars.nic.in/D2/D02adm/OM%20dated%202%207%2097BsMyq, which was last accessed on 24 May 2024.



(a) CDF of Webster-Sainte-Laguë's Staircase Roster ($\Lambda^{\mathtt{IN}}$)



(b) CDF for the Indian Roster

Fig. 4 Distribution of seats

The evaluation conducted based on the properties discussed in our paper $\Phi^{\mathtt{IN}}$ is far from ideal. It is easy to verify that $\Phi^{\mathtt{IN}}$ is exact, anonymous, house-monotonic, and concatenation invariant. It is responsive to $\Lambda^{\mathtt{IN}}$; however, it is not responsive, consistent, or balanced in general. It does not minimize the nonuniformity index, does not minimize inequality and does not stay near the quota. Last, it cannot be derived from a divisor method (see Appendix 1 for proofs of these claims).

6 Conclusion

In this study, we addressed the problem of constructing rosters for affirmative action in India, which involves solving a series of connected apportionment problems. We focused on a large class of divisor methods, identifying the Webster–Sainte-Laguë method as uniquely satisfying essential practical and fairness criteria. Our comparative analysis of an existing Indian roster with one derived from the Webster–Sainte-Laguë method highlights the latter's advantages.

Appendix 1 Proofs

Let $x_j^t := |\{s \in R_n^{-1}(j) \mid s \leq t\}|$ be the number of seats given to category j until point t. Note that $\sum_j x_j^t = t$ for $t \in \{1, \cdots, n\}$. Let n_j be the total number of seats in the roster for category j, that is, $n_j = x_j^n = \alpha_j n$.

With these definitions, the distribution of seats for category j at point t is

$$F_j(t) = \frac{x_j^t}{n_j}$$

and the location at the staircase at point t is

$$\mathbf{x^t} = (x_1^t, \cdots, x_j^t, \cdots, x_n^t).$$

Proof of Theorem 2

Proof $\mathbf{x}^t \in \Phi^d(\mathbf{q}^t)$ if and only if $x_j^t \geq 0$ for all $j \in \mathcal{C}$, $\sum_{j \in \mathcal{C}} x_j^t = t$, and

$$\min_{x_i^t > 0} \frac{q_i^t}{d\left(x_i^t - 1\right)} \geq \max_{x_j^t \geq 0} \frac{q_j^t}{d\left(x_j^t\right)}.$$

Equivalently,

$$\frac{d\left(x_i^t - 1\right)}{d\left(x_j^t\right)} \leq \frac{\alpha_i}{\alpha_j} \leq \frac{d\left(x_i^t\right)}{d\left(x_j^t - 1\right)} \quad \text{if } x_i^t, x_j^t > 0, \tag{1}$$

$$\frac{\alpha_i}{\alpha_j} \leq \frac{d(0)}{d(x_j^t - 1)} \quad \text{if } x_i^t = 0, \quad x_j^t > 0, \tag{2}$$

and

$$\frac{d(x_i^t - 1)}{d(0)} \le \frac{\alpha_i}{\alpha_j} \quad \text{if } x_i^t > 0, \quad x_j^t = 0. \tag{3}$$

Next, since divisor methods are exact (as defined in Section 3), $\mathbf{x}^s \in \Phi^d(\mathbf{q}^s)$ we have,

$$\frac{x_i^s}{x_j^s} = \frac{\alpha_i}{\alpha_j}. \tag{4}$$

Recall the following fact: adding a to the numerator and b to the denominator moves the resultant fraction closer to the fraction a/b. If $x/y < a/b$, moving the starting fraction close to a/b will make it bigger. If $x/y > a/b$, moving the starting fraction close to $\frac{a}{b}$ will make it smaller.

Adding x_i^s to numerators and x_j^s to denominators in Eq. (1)–(3) therefore does not alter the inequalities and gives,

$$\frac{d\left(x_i^t - 1\right) + x_i^s}{d\left(x_j^t\right) + x_j^s} \le \frac{\alpha_i}{\alpha_j} \le \frac{d\left(x_i^t\right) + x_i^s}{d\left(x_j^t - 1\right) + x_j^s} \quad \text{if } x_i^t, x_j^t > 0, \tag{5}$$

$$\frac{\alpha_i}{\alpha_j} \le \frac{d(0) + x_i^s}{d(x_j^t - 1) + x_j^s} \quad \text{if } x_i^t = 0, \quad x_j^t > 0, \tag{6}$$

and

$$\frac{d(x_i^t - 1) + x_i^s}{d(0) + x_j^s} \le \frac{\alpha_i}{\alpha_j} \quad \text{if } x_i^t > 0, \quad x_j^t = 0. \tag{7}$$

If $d(x_i^t - 1) + x_i^s = d(x_i^t + x_i^s - 1)$, $d(x_i^t) + x_i^s = d(x_i^t + x_i^s)$, and $d(0) + x_j^s = d(x_j^s)$, then we have that $\mathbf{x}^t + \mathbf{x}^s \in \Phi^d\left(\mathbf{q}^{t+s}\right)$, that is Φ^d is concatenation invariant. In particular, all parametric methods $d(a) = a + \delta$, where $0 \le \delta \le 1$, for all integer $a \ge 0$, are concatenation invariant. □

Proof of Theorem 3

Outline This proof contains three parts. First, we will reduce $\mathrm{DI}(\mathbf{x^t})$ to a more manageable expression. Second, in Lemma 1, we will show that if $\mathbf{x^t}$ is optimal in that it minimizes the nonuniformity index, then for any pair of i, j with $x_i^t > 0$, we have,

$$\frac{x_i^t - 0.5}{\alpha_i} \le \frac{x_j^t + 0.5}{\alpha_j}.$$

Therefore, $\mathbf{x^t}$ optimal implies

$$\min_{x_i^t > 0} \frac{\alpha_i}{x_i^t - 0.5} \ge \max_{x_j^t \ge 0} \frac{\alpha_j}{x_j^t + 0.5},$$

which is the min-max inequality that characterizes Webster-Sainte-Laguë apportionments. Third, in Lemma 2, we will prove that Webster-Sainte-Laguë method is the unique divisor method that minimizes the nonuniformity index by showing that the set of minimizers of the nonuniformity index and the Webster-Sainte-Laguë apportionments are the same.

We denote by $\mathrm{DI}(\mathbf{x^t})$ weighted distance between the distribution of seats and the uniform distribution at point t; that is,

$$\mathrm{DI}(\mathbf{x^t}) = \sum_j \alpha_j (F_j(t) - U(t))^2.$$

Note that,

$$\begin{aligned}
\mathrm{DI}(\mathbf{x^t}) &= \sum_j \alpha_j (F_j(t) - U(t))^2 \\
&= \sum_j \alpha_j \Big(\frac{x_j^t}{\alpha_j n} - \frac{t}{n}\Big)^2 \\
&= \frac{1}{n^2} \sum_j \frac{x_j^{t2}}{\alpha_j} - \frac{1}{n^2} \sum_j 2x_j^t t + \frac{1}{n^2} \sum_j \alpha_j t^2 \\
&= \frac{1}{n^2} \sum_j \frac{x_j^{t2}}{\alpha_j} - \frac{2t^2}{n^2} + \frac{t^2}{n^2} \\
&= \frac{1}{n^2} \sum_j \frac{x_j^{t2}}{\alpha_j} - \frac{t^2}{n^2}.
\end{aligned}$$

Therefore, minimizing $\mathrm{DI}(\mathbf{x^t})$ is equivalent to minimizing $\sum_j \frac{x_j^{t2}}{\alpha_j}$.

Lemma 1 *For any* $\mathbf{x^t} = (x_1^t, \cdots, x_j^t, \cdots, x_m^t)$, *for any pair of* i, j *with* $x_i^t > 0$,

$$\mathrm{DI}(\mathbf{x^t}) \leq \mathrm{DI}(\mathbf{x^t} + e_j - e_i) \iff \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}.$$

Proof Using the fact that $\mathrm{DI}(\mathbf{x^t}) = \frac{1}{n^2} \sum_j \frac{x_j^{t2}}{\alpha_j} - \frac{t^2}{n^2}$,

$$\begin{aligned}
\mathrm{DI}(\mathbf{x^t}) \leq \mathrm{DI}(\mathbf{x^t} + e_j - e_i) &\iff \frac{x_i^{t2}}{\alpha_i} + \frac{x_j^{t2}}{\alpha_j} \leq \frac{(x_i^t - 1)^2}{\alpha_i} + \frac{(x_j^t + 1)^2}{\alpha_j} \\
&\iff \frac{x_i^{t2}}{\alpha_i} - \frac{(x_i^t - 1)^2}{\alpha_i} \leq \frac{(x_j^t + 1)^2}{\alpha_j} - \frac{x_j^{t2}}{\alpha_j} \\
&\iff \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}.
\end{aligned}$$

□

Lemma 2 *The following two sets are equivalent*:

$$\arg\min_{\mathbf{x^t}} \mathrm{DI}(\mathbf{x^t}) \text{ s.t. } \sum_j x_j^t = t \text{ and } \mathbf{x^t} \geq 0 \text{ integer} \tag{8}$$

$$\left\{ \mathbf{x^t} \mid \text{ for any } i, j \text{ with } x_i^t > 0, \frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j}; \sum_j x_j^t = t \text{ and } \mathbf{x^t} \geq 0 \text{ integer} \right\} \tag{9}$$

Proof Lemma 1 implies that if $\mathbf{x^t}$ minimizes $\mathrm{DI}(\mathbf{x^t})$ then the following inequalities hold:

$$\frac{x_i^t - 0.5}{\alpha_i} \leq \frac{x_j^t + 0.5}{\alpha_j} \text{ for any } i, j \text{ with } x_i^t > 0.$$

Therefore, if $\mathbf{x^t}$ is in the former set (8), then it is also in the latter set (9).

Suppose $\mathbf{y^t}$ is in the latter set (9) but not in the former set (8); that is,

$$\frac{y_i^t - 0.5}{\alpha_i} \leq \frac{y_j^t + 0.5}{\alpha_j} \text{ for any } i, j \text{ with } y_i^t > 0.$$

We denote by $H = \{j | x_j^t > y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has more number of seats. We denote by $L = \{j | x_j^t < y_j^t\}$ the set of categories in $\mathbf{x^t}$ that has less number of seats. We denote by $h_j := x_j^t - y_j^t$ for $j \in H$. We denote by $l_j := y_j^t - x_j^t$ for $j \in L$. Since $\sum_j y_j^t = \sum_j x_j^t = t$, we have $\sum_{j \in H} h_j = \sum_{j \in L} l_j > 0$.

Using the inequalities for $\mathbf{x^t}$ and $\mathbf{y^t}$ for $i \in L$ and $j \in H$, we have

$$\frac{2y_i^t - l_j}{\alpha_i} \leq \frac{2y_j^t + h_j}{\alpha_j} \text{ for any } i \in L \text{ and } j \in H.$$

If we calculate the summation of such inequities, we find that

$$\sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \leq \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j}.$$

Note that,

$$\begin{aligned}
\sum_j \frac{x_j^{t2}}{\alpha_j} - \sum_j \frac{y_j^{t2}}{\alpha_j} &= \sum_j \left(\frac{x_j^{t2}}{\alpha_j} - \frac{y_j^{t2}}{\alpha_j} \right) \\
&= \sum_{j \in H} \left(\frac{x_j^{t2}}{\alpha_j} - \frac{y_j^{t2}}{\alpha_j} \right) - \sum_{j \in L} \left(\frac{y_j^{t2}}{\alpha_j} - \frac{x_j^{t2}}{\alpha_j} \right) \\
&= \sum_{j \in H} \frac{h_j(2y_j^t + h_j)}{\alpha_j} - \sum_{j \in L} \frac{l_j(2y_j^t - l_j)}{\alpha_j} \\
&\geq 0.
\end{aligned}$$

This contradicts the assumption that $\mathbf{y^t}$ does not belong to the former set (8); that is, if a $\mathbf{y^t}$ is in the latter set (9) then $\mathbf{y^t}$ must be in the former set (8). □

Proof of Theorem 3 Lemma 2 shows that the set of minimizers of the nonuniformity index and the set of Webster-Sainte-Laguë apportionments are the same. Therefore, Webster-Sainte-Laguë method is the unique divisor method that minimizes the nonuniformity index. ☐

Proof of Theorem 1

Proof

$$
\begin{aligned}
\mathrm{DI}(\mathbf{x^t}) &= \sum_{j\in\mathcal{C}} \alpha_j (F_j(t) - U(t))^2 \\
&= \sum_{j\in\mathcal{C}} \alpha_j \left(\frac{x_j^t}{\alpha_j n} - \frac{t}{n} \right)^2 \\
&= \sum_{j\in\mathcal{C}} \alpha_j \left(\frac{x_j^t - \alpha_j t}{\alpha_j n} \right)^2 \\
&= \frac{t}{n^2} \sum_{j\in\mathcal{C}} \frac{(x_j^t - \alpha_j t)^2}{\alpha_j t} \\
&= \frac{t}{n^2} \mathrm{SLI}(\mathbf{x^t}).
\end{aligned}
$$

☐

Claims about $\Phi^{\mathtt{IN}}$ in Section 5.2

Claim 1 $\Phi^{\mathtt{IN}}$ *does not minimize the nonuniformity index.*

Proof $\Phi^{\mathtt{IN}}$ does not minimize the nonuniformity index as for $\Lambda^{\mathtt{IN}}$, $DI(\mathbf{x^3}) > DI(\mathbf{x^3} + \mathbf{e_{SC}} - \mathbf{e_{UR}})$. ☐

Claim 2 $\Phi^{\mathtt{IN}}$ *does not minimize inequality.*

Proof $\Phi^{\mathtt{IN}}$ does not minimize inequality as for $\Lambda^{\mathtt{IN}}$,

$$
\left| \frac{x_{SC}^3}{n_{SC}} - \frac{x_{UR}^3}{n_{UR}} \right| = \left| \frac{0}{30} - \frac{3}{81} \right| > \left| \frac{1}{30} - \frac{2}{81} \right| = \left| \frac{x_{SC}^3 + 1}{n_{SC}} - \frac{x_{UR}^3 - 1}{n_{UR}} \right|.
$$

☐

Claim 3 $\Phi^{\mathtt{IN}}$ *does not satisfy stays near quota.*

Proof $\Phi^{\mathtt{IN}}$ does not satisfy stays near quota as for $\Lambda^{\mathtt{IN}}$,

$$
\left| (x_{UR}^2 - 1) - q_{UR}^2 \right| < \left| x_{UR}^2 - q_{UR}^2 \right| \text{ and } \left| (x_{OBC}^2 + 1) - q_{OBC}^2 \right| < \left| x_{OBC}^2 - q_{OBC}^2 \right|.
$$

☐

Claim 4 $\Phi^{\mathtt{IN}}$ *cannot be derived from a divisor method.*

Proof $\Phi^{\mathtt{IN}}$ cannot be derived from a divisor method, as for $\Lambda^{\mathtt{IN}}$ and $t = 36$,

	SC	*ST*	*OBC*	*EWS*	*UR*
q	5.4	2.7	9.72	3.6	14.58
x	5	2	9	3	17

For a divisor method, we need

$$\min_i \frac{q_i}{d(x_i - 1)} \geq \max_j \frac{q_j}{d(x_j)} \quad \text{where } d(k) \in [k, k+1]. \tag{10}$$

However,

$$\max_j \frac{q_j}{d(x_j)} \geq \max_j \frac{q_j}{x_j + 1} = \max\left\{\frac{5.4}{6}, \frac{2.7}{3}, \frac{9.72}{10}, \frac{3.6}{4}, \frac{14.58}{18}\right\} = 0.972. \tag{11}$$

And,

$$\min_i \frac{q_i}{d(x_i - 1)} \leq \min_i \frac{q_i}{x_i - 1} = \min\left\{\frac{5.4}{4}, \frac{2.7}{1}, \frac{9.72}{8}, \frac{3.6}{2}, \frac{14.58}{16}\right\} = \frac{14.58}{16} \approx 0.91125 < 0.972. \tag{12}$$

Therefore, for all $d(k)$ such that $d(k) \in [k, k+1]$,

$$\max_j \frac{q_j}{d(x_j)} \geq \max_j \frac{q_j}{x_j + 1} = 0.972 > 0.91125 = \min_i \frac{q_i}{x_i - 1} \geq \min_i \frac{q_i}{d(x_i - 1)}. \tag{13}$$

Equation (13) implies that Eq. (10) cannot be satisfied; therefore, $\Phi^{\mathtt{IN}}$ cannot be derived from a divisor method.

□

Appendix 2 Rosters for $\Lambda^{\mathtt{IN}}$

No.	SC@15%	ST@7.5%	OBC@27%	EWS@10%	Indian	Webster-Sainte-Laguë
1	0.15	0.08	0.27	0.1	UR	UR
2	0.3	0.15	0.54	0.2	UR	OBC-1
3	0.45	0.22	0.81	0.3	UR	SC-1
4	0.6	0.3	1.08	0.4	OBC-1	UR
5	0.75	0.38	1.35	0.5	UR	EWS-1
6	0.9	0.45	1.62	0.6	UR	OBC-2
7	1.05	0.52	1.89	0.7	SC-1	UR

No.	SC@15%	ST@7.5%	OBC@27%	EWS@10%	Indian	Webster-Sainte-Laguë
8	1.2	0.6	2.16	0.8	OBC-2	ST-1
9	1.35	0.68	2.43	0.9	UR	UR
10	1.5	0.75	2.7	1	EWS-1	OBC-3
11	1.65	0.82	2.97	1.1	UR	SC-2
12	1.8	0.9	3.24	1.2	OBC-3	UR
13	1.95	0.98	3.51	1.3	UR	OBC-4
14	2.1	1.05	3.78	1.4	ST-1	UR
15	2.25	1.12	4.05	1.5	SC-2	EWS-2
16	2.4	1.2	4.32	1.6	OBC-4	UR
17	2.55	1.27	4.59	1.7	UR	OBC-5
18	2.7	1.35	4.86	1.8	UR	SC-3
19	2.85	1.42	5.13	1.9	OBC-5	UR
20	3	1.5	5.4	2	SC-3	ST-2
21	3.15	1.58	5.67	2.1	EWS-2	OBC-6
22	3.3	1.65	5.94	2.2	UR	UR
23	3.45	1.72	6.21	2.3	OBC-6	SC-4
24	3.6	1.8	6.48	2.4	UR	UR
25	3.75	1.88	6.75	2.5	UR	OBC-7
26	3.9	1.95	7.02	2.6	OBC-7	EWS-3
27	4.05	2.02	7.29	2.7	SC-4	UR
28	4.2	2.1	7.56	2.8	ST-2	OBC-8
29	4.35	2.17	7.83	2.9	UR	UR
30	4.5	2.25	8.1	3	OBC-8	SC-5
31	4.65	2.32	8.37	3.1	EWS-3	UR
32	4.8	2.4	8.64	3.2	UR	OBC-9
33	4.95	2.48	8.91	3.3	UR	UR
34	5.1	2.55	9.18	3.4	OBC-9	ST-3
35	5.25	2.62	9.45	3.5	SC-5	EWS-4
36	5.4	2.7	9.72	3.6	UR	OBC-10
37	5.55	2.78	9.99	3.7	UR	UR
38	5.7	2.85	10.26	3.8	OBC-10	SC-6
39	5.85	2.92	10.53	3.9	UR	UR
40	6	3	10.8	4	ST-3	OBC-11
41	6.15	3.08	11.07	4.1	SC-6	UR
42	6.3	3.15	11.34	4.2	OBC-11	OBC-12
43	6.45	3.22	11.61	4.3	EWS-4	UR
44	6.6	3.3	11.88	4.4	UR	SC-7
45	6.75	3.38	12.15	4.5	OBC-12	EWS-5
46	6.9	3.45	12.42	4.6	UR	UR
47	7.05	3.52	12.69	4.7	SC-7	OBC-13

48	7.2	3.6	12.96	4.8	UR	ST-4
49	7.35	3.68	13.23	4.9	OBC-13	UR
50	7.5	3.75	13.5	5	EWS-5	OBC-14
51	7.65	3.82	13.77	5.1	UR	SC-8
52	7.8	3.9	14.04	5.2	OBC-14	UR
53	7.95	3.97	14.31	5.3	UR	UR
54	8.1	4.05	14.58	5.4	SC-8	OBC-15
55	8.25	4.12	14.85	5.5	ST-4	EWS-6
56	8.4	4.2	15.12	5.6	OBC-15	UR
57	8.55	4.27	15.39	5.7	UR	SC-9
58	8.7	4.35	15.66	5.8	UR	OBC-16
59	8.85	4.42	15.93	5.9	UR	UR
60	9	4.5	16.2	6	OBC-16	ST-5
61	9.15	4.58	16.47	6.1	SC-9	UR
62	9.3	4.65	16.74	6.2	EWS-6	OBC-17
63	9.45	4.72	17.01	6.3	OBC-17	UR
64	9.6	4.8	17.28	6.4	UR	SC-10
65	9.75	4.88	17.55	6.5	UR	OBC-18
66	9.9	4.95	17.82	6.6	UR	EWS-7
67	10.05	5.02	18.09	6.7	OBC-18	UR
68	10.2	5.1	18.36	6.8	SC-10	UR
69	10.35	5.18	18.63	6.9	ST-5	OBC-19
70	10.5	5.25	18.9	7	EWS-7	SC-11
71	10.65	5.32	19.17	7.1	OBC-19	UR
72	10.8	5.4	19.44	7.2	UR	OBC-20
73	10.95	5.48	19.71	7.3	UR	UR
74	11.1	5.55	19.98	7.4	SC-11	ST-6
75	11.25	5.62	20.25	7.5	OBC-20	EWS-8
76	11.4	5.7	20.52	7.6	UR	UR
77	11.55	5.78	20.79	7.7	UR	OBC-21
78	11.7	5.85	21.06	7.8	OBC-21	SC-12
79	11.85	5.92	21.33	7.9	UR	UR
80	12	6	21.6	8	ST-6	OBC-22
81	12.15	6.08	21.87	8.1	SC-12	UR
82	12.3	6.15	22.14	8.2	OBC-22	UR
83	12.45	6.22	22.41	8.3	EWS-8	OBC-23
84	12.6	6.3	22.68	8.4	UR	SC-13
85	12.75	6.38	22.95	8.5	UR	EWS-9
86	12.9	6.45	23.22	8.6	OBC-23	UR
87	13.05	6.52	23.49	8.7	SC-13	ST-7
88	13.2	6.6	23.76	8.8	UR	OBC-24
89	13.35	6.68	24.03	8.9	OBC-24	UR
90	13.5	6.75	24.3	9	EWS-9	SC-14
91	13.65	6.82	24.57	9.1	UR	UR
92	13.8	6.9	24.84	9.2	UR	OBC-25
93	13.95	6.98	25.11	9.3	OBC-25	UR

94	14.1	7.05	25.38	9.4	SC-14	OBC-26
95	14.25	7.12	25.65	9.5	ST-7	EWS-10
96	14.4	7.2	25.92	9.6	UR	UR
97	14.55	7.28	26.19	9.7	OBC-26	SC-15
98	14.7	7.35	26.46	9.8	EWS-10	UR
99	14.85	7.42	26.73	9.9	SC-15	OBC-27
100	15	7.5	27	10	OBC-27	UR
101	15.15	7.57	27.27	10.1	UR	ST-8
102	15.3	7.65	27.54	10.2	UR	OBC-28
103	15.45	7.72	27.81	10.3	UR	UR
104	15.6	7.8	28.08	10.4	OBC-28	SC-16
105	15.75	7.88	28.35	10.5	UR	UR
106	15.9	7.95	28.62	10.6	UR	EWS-11
107	16.05	8.02	28.89	10.7	SC-16	OBC-29
108	16.2	8.1	29.16	10.8	ST-8	UR
109	16.35	8.17	29.43	10.9	OBC-29	OBC-30
110	16.5	8.25	29.7	11	EWS-11	UR
111	16.65	8.32	29.97	11.1	UR	SC-17
112	16.8	8.4	30.24	11.2	OBC-30	UR
113	16.95	8.48	30.51	11.3	UR	OBC-31
114	17.1	8.55	30.78	11.4	SC-17	ST-9
115	17.25	8.62	31.05	11.5	OBC-31	UR
116	17.4	8.7	31.32	11.6	UR	EWS-12
117	17.55	8.78	31.59	11.7	UR	OBC-32
118	17.7	8.85	31.86	11.8	UR	SC-18
119	17.85	8.92	32.13	11.9	OBC-32	UR
120	18	9	32.4	12	ST-9	UR
121	18.15	9.07	32.67	12.1	SC-18	OBC-33
122	18.3	9.15	32.94	12.2	EWS-12	UR
123	18.45	9.22	33.21	12.3	OBC-33	SC-19
124	18.6	9.3	33.48	12.4	UR	OBC-34
125	18.75	9.38	33.75	12.5	UR	UR
126	18.9	9.45	34.02	12.6	OBC-34	EWS-13
127	19.05	9.52	34.29	12.7	SC-19	ST-10
128	19.2	9.6	34.56	12.8	UR	UR
129	19.35	9.67	34.83	12.9	UR	OBC-35
130	19.5	9.75	35.1	13	OBC-35	UR
131	19.65	9.82	35.37	13.1	EWS-13	SC-20
132	19.8	9.9	35.64	13.2	UR	OBC-36
133	19.95	9.98	35.91	13.3	UR	UR
134	20.1	10.05	36.18	13.4	OBC-36	UR
135	20.25	10.12	36.45	13.5	SC-20	EWS-14
136	20.4	10.2	36.72	13.6	ST-10	OBC-37
137	20.55	10.28	36.99	13.7	UR	SC-21
138	20.7	10.35	37.26	13.8	OBC-37	UR
139	20.85	10.42	37.53	13.9	UR	OBC-38

140	21	10.5	37.8	14	SC-21	UR
141	21.15	10.58	38.07	14.1	OBC-38	ST-11
142	21.3	10.65	38.34	14.2	EWS-14	UR
143	21.45	10.72	38.61	14.3	UR	OBC-39
144	21.6	10.8	38.88	14.4	UR	SC-22
145	21.75	10.88	39.15	14.5	OBC-39	UR
146	21.9	10.95	39.42	14.6	UR	EWS-15
147	22.05	11.02	39.69	14.7	SC-22	OBC-40
148	22.2	11.1	39.96	14.8	ST-11	UR
149	22.35	11.18	40.23	14.9	OBC-40	UR
150	22.5	11.25	40.5	15	EWS-15	OBC-41
151	22.65	11.32	40.77	15.1	UR	SC-23
152	22.8	11.4	41.04	15.2	OBC-41	UR
153	22.95	11.48	41.31	15.3	UR	ST-12
154	23.1	11.55	41.58	15.4	SC-23	OBC-42
155	23.25	11.62	41.85	15.5	UR	UR
156	23.4	11.7	42.12	15.6	OBC-42	EWS-16
157	23.55	11.78	42.39	15.7	UR	SC-24
158	23.7	11.85	42.66	15.8	UR	UR
159	23.85	11.92	42.93	15.9	UR	OBC-43
160	24	12	43.2	16	ST-12	UR
161	24.15	12.08	43.47	16.1	OBC-43	OBC-44
162	24.3	12.15	43.74	16.2	SC-24	UR
163	24.45	12.22	44.01	16.3	OBC-44	SC-25
164	24.6	12.3	44.28	16.4	EWS-16	UR
165	24.75	12.38	44.55	16.5	UR	OBC-45
166	24.9	12.45	44.82	16.6	UR	EWS-17
167	25.05	12.52	45.09	16.7	OBC-45	UR
168	25.2	12.6	45.36	16.8	SC-25	ST-13
169	25.35	12.68	45.63	16.9	UR	OBC-46
170	25.5	12.75	45.9	17	EWS-17	UR
171	25.65	12.82	46.17	17.1	OBC-46	SC-26
172	25.8	12.9	46.44	17.2	UR	UR
173	25.95	12.98	46.71	17.3	UR	OBC-47
174	26.1	13.05	46.98	17.4	SC-26	UR
175	26.25	13.12	47.25	17.5	ST-13	EWS-18
176	26.4	13.2	47.52	17.6	OBC-47	OBC-48
177	26.55	13.28	47.79	17.7	UR	UR
178	26.7	13.35	48.06	17.8	OBC-48	SC-27
179	26.85	13.42	48.33	17.9	UR	UR
180	27	13.5	48.6	18	SC-27	OBC-49
181	27.15	13.58	48.87	18.1	EWS-18	ST-14
182	27.3	13.65	49.14	18.2	OBC-49	UR
183	27.45	13.72	49.41	18.3	UR	OBC-50
184	27.6	13.8	49.68	18.4	UR	SC-28
185	27.75	13.88	49.95	18.5	UR	UR

186	27.9	13.95	50.22	18.6	OBC-50	EWS-19
187	28.05	14.02	50.49	18.7	SC-28	UR
188	28.2	14.1	50.76	18.8	ST-14	OBC-51
189	28.35	14.18	51.03	18.9	OBC-51	UR
190	28.5	14.25	51.3	19	EWS-19	SC-29
191	28.65	14.32	51.57	19.1	UR	OBC-52
192	28.8	14.4	51.84	19.2	UR	UR
193	28.95	14.48	52.11	19.3	OBC-52	ST-15
194	29.1	14.55	52.38	19.4	SC-29	UR
195	29.25	14.62	52.65	19.5	UR	OBC-53
196	29.4	14.7	52.92	19.6	EWS-20	EWS-20
197	29.55	14.77	53.19	19.7	OBC-53	UR
198	29.7	14.85	53.46	19.8	ST-15	SC-30
199	29.85	14.92	53.73	19.9	SC-30	OBC-54
200	30	15	54	20	OBC-54	UR

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